1. State space postulate

- The state of a system is described by a unit vector in a Hilbert space.

\[
\begin{pmatrix}
\alpha_0 \\
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{pmatrix}
\]

where

\[|\alpha_0|^2 + |\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 = 1\]

\[\alpha_0|00\rangle + \alpha_1|01\rangle + \alpha_2|10\rangle + \alpha_3|11\rangle\]

complex amplitude
1. State space postulate

Representations of a single qubit

\[ |\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \]

\[ |\psi\rangle = \alpha|0\rangle + \beta|1\rangle \]

\[ |\psi\rangle = \cos \frac{\theta}{2}|0\rangle + e^{i \phi} \sin \frac{\theta}{2}|1\rangle \]
2. Evolution postulate

A physical system changes in time, and so the state vector $|\psi\rangle$ of a system will actually by a function of time.

- The time-evolution of the state of a closed quantum system is described by a unitary operator. That is, for any evolution of the closed system there exists a unitary operator $U$ such that if the initial state of the system is $|\psi\rangle$, then after the evolution of the state of the system will be

$$|\psi_2\rangle = U|\psi_1\rangle$$

The evolution of the state vector of a closed system is linear

$$U \sum_i \alpha_i |\psi_i\rangle = \sum_i \alpha_i U |\psi_i\rangle$$
2. Evolution postulate

• The only linear operators that preserve such norms of vectors are the unitary operators

• A linear operator is specified completely by its action on a basis

• Example: Pauli operators

\[ \sigma_0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_x = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

\[ \sigma_y = Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

• Unitary is implied by the continuous time evolution of a closed quantum mechanical system as described by the Schrödinger equation

\[ i\hbar \frac{\psi'(t)}{dt} = H(t)\psi(t) \]

\[ \psi(t_2) = e^{-i\hbar H(t_2-t_1)}\psi(t_1) \]

Unitary operator
3. Composition of system postulate

- When two physical systems are treated as one combined system, the state space of the combined physical system is the tensor product space $H_1 \otimes H_2$ of the state space $H_1$, $H_2$ of the component subsystems. In the first system is the state $|\psi_1\rangle$ and the second system in the state $|\psi_2\rangle$, then the state of the combined system is $|\psi_1\rangle \otimes |\psi_2\rangle$.

- If two qubits are prepared independently, and kept isolated, then each qubit forms a closed system, and the state can be written in the product form.

- If the qubits are allowed to interact, then the closed system includes both qubits together, and it may not be possible to write the state in the product form.
4. Measurement postulate

The evolution of the state of a system during a measurement is not unitary, and an additional postulate is needed to describe measurement

- For a given orthonormal basis $B = \{ |\psi_i\rangle \}$ of a state space $H_A$ for a given system $A$, it is possible to perform a Von Neumann measurements on system $H_A$ with respect to the basis $B$ that, given a state

$$|\psi\rangle = \sum_i \alpha_i |\phi_i\rangle$$

outputs a label $i$ with probability $|\alpha_i|^2$ and leaves the system in state $|\phi_i\rangle$
4. Measurement postulate

\[ |\psi\rangle = \sum_i \alpha_i |\phi_i\rangle \]

\[ \alpha_i = \langle \phi_i | \psi \rangle \quad p_i = |\alpha_i|^2 = \alpha_i^* \alpha_i = \langle \psi | \phi_i \rangle \langle \phi_i | \psi \rangle \]

Question: What are the result of measuring the first qubit?

\[ |\psi\rangle = \sqrt{\frac{1}{11}} |0\rangle |0\rangle + \sqrt{\frac{5}{11}} |0\rangle |1\rangle + \sqrt{\frac{2}{11}} |1\rangle |0\rangle + \sqrt{\frac{3}{11}} |1\rangle |1\rangle \]

Two states \( |\psi\rangle \) and \( e^{i\theta} |\psi\rangle \) (differing only by a global phase) are equivalent

\[ e^{i\theta} |\psi\rangle = \sum_i \alpha_i e^{i\theta} |\psi_i\rangle \]

\[ p_i = \alpha_i^* e^{-i\theta} \alpha_i e^{i\theta} = |\alpha_i|^2 \]