EN2912C: Future Directions in Computing
Lecture 16: Linear Algebra Refresher

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Dirac notation

- In Dirac notation, a symbol identifying a vector is written inside a 'ket': $|a\rangle$
- The dual vector (Hermitian conjugate) for *a* is defined with a 'bra' written as $\langle a|$
- Let's think of a Hilbert space as a finite dimensional complex vector space

$$\underbrace{|00\ldots00\rangle}_{n} \leftrightarrow \begin{pmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \\ 0 \end{pmatrix} 2^n \quad |00\ldots01\rangle \leftrightarrow \begin{pmatrix} 0 \\ 1 \\ . \\ . \\ . \\ 0 \\ 0 \end{pmatrix} \quad \ldots \quad |11\ldots11\rangle \leftrightarrow \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ 1 \end{pmatrix}$$

2^n basis vectors for a 2^n Hilbert space

Dot product

The inner (dot) product of two vectors v and w: $\langle \mathbf{v} | \mathbf{w} \rangle$

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = (v_1^* v_2^* \ldots v_n^*) \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \sum_{i=1}^{n} v_i^* w_i$$

The norm of vector $|\psi\rangle$ is equal to $\| |\psi\rangle \| = \sqrt{\langle \psi | \psi \rangle}$

Tensor product

A is 2 × 2 matrix and B is 3 × 2 matrix

$$A \otimes B = \begin{bmatrix} a_{11}\,\mathrm{B} & a_{12}\,\mathrm{B} \\ a_{21}\,\mathrm{B} & a_{22}\,\mathrm{B} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{11}b_{31} & a_{11}b_{32} & a_{12}b_{31} & a_{12}b_{32} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \\ a_{21}b_{31} & a_{21}b_{32} & a_{22}b_{31} & a_{22}b_{32} \end{bmatrix}.$$

Example: Find the tensor product of $\alpha_0|0\rangle + \alpha_1|1\rangle$ and $\beta_0|0\rangle + \beta_1|1\rangle$

In Dirac notation:

$$(\alpha_0|0\rangle + \alpha_1|1\rangle) \otimes (\beta_0|0\rangle + \beta_1|1\rangle) = \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle$$

In matrix notation:

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \otimes \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{pmatrix}$$

Example

Example: Find the dot product of the following two vectors once using matrix notation and once using Dirac notation

$$|\psi\rangle = \sqrt{\frac{2}{3}}|01\rangle + \frac{i}{\sqrt{3}}|11\rangle \qquad |\phi\rangle = \sqrt{\frac{1}{2}}|10\rangle + \sqrt{\frac{1}{2}}|11\rangle$$

Operators

- Unitary operator

$$U^\dagger = U^{-1}$$

- Hermitian operator

$$T^\dagger = T$$

- Eigenvector

$$T|\psi\rangle = c|\psi\rangle$$

- Matrix trace

$$Tr(A) = \sum_{b_n} \langle b_n | A | b_n \rangle$$

Spectral theorem

- If *T* is normal (e.g., hermitian or unitary), then there is a unitary matrix P such that

$$T = P\Lambda P^\dagger$$

where Λ are the eigenvlues of *T*